New approach of the hairy ball theorem

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New approach of the hairy ball theorem

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Abstract

In this paper, we establish an equivalent version of the hairy ball theorem in the form of a fixed point theorem. By using a version of Mas-Colell theorem [6] and by applying homotopy and approximation methods, we obtain our main result.

Keywords: Hairy ball theorem, fixed point theorems, approximation methods, homotopy, topological degree, connected components.

1 Introduction

The aim of this paper is to establish and prove the following fixed point theorem equivalent to the hairy ball theorem.

If $f : S \to S$ is continuous and satisfying for any $x \in S, f(x).x \geq \frac{1}{2}$, then it possesses a fixed point.
The proof of this theorem is based on constructing of an explicit continuous homotopy F between a nontrivial function α and the function f. A component starting from fixed points of α will lead us to a fixed point of f. Yet, using a result of Mas-Colell [6], we prove the result for a twice continuous differentiable function. Our point is that the existence of a smooth path can easily be implented on computers. Then, using approximation technics, we recover all fixed points of f.
The paper is organized in the following way. Section 2 contains preliminaries and notations. The equivalent fixed points theorems are set in section 3. The proof of the main result is given in section 4.

2 Preliminaries and notations

Throughout this paper, we shall use the following notations and definitions.
Let $S = S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ be the unit $n-$sphere. For any $0 < r < 1$, we denote by $B(S, r) = \{x \in \mathbb{R}^{n+1} : 1 - r \leq \|x\| \leq 1 + r\}$.
For a subset $X \subset \mathbb{R}^n$, we denote by $\overline{X}$ the closure of X, by X^c the complement of X, by $\text{int}(X)$ the interior of X, and by $\partial X = \overline{X} \setminus \text{int}(X)$ the boundary of X. We denote by $x_0 = (0, 0, \cdots, 1)$ and $-x_0 = (0, 0, \cdots, -1)$, respectively the north and south pole of S and by 'deg' the classical topological degree.

Definition 1. *We denote by $\mathscr{H}$ the set of continuous functions $F : [0, 1] \times S \to S$ such that $F(0, x_0) = x_0$. For a continuous function $F : [0, 1] \times S \to S$, we denote by*

1. $C_F := \{(t, x) \in [0, 1] \times S : F(t, x) = x\}$.
2. *The translation $\overline{F}$ of F is defined by $\overline{F}(t, x) = F(t, x) - x$.*

For getting our main result, we need the following topological degree properties.

Proposition 1. *Let Ω be a bounded open set of $\mathbb{R}^m$, $f : \overline{\Omega} \longrightarrow \mathbb{R}^m$ be a continuous function and $y \in \mathbb{R}^m$ such that $y \notin f(\partial\Omega)$. Then, we have*

i. $\deg(., \Omega, y)$ *is constant in* $\{g \in C(\overline{\Omega}) \setminus \|g - f\| < r\}$ *where* $r = d(y, f(\partial\Omega))$.

ii. *Let Ω_1 be an open set of Ω. If $y \notin f(\overline{\Omega} \setminus \Omega_1)$, then* $\deg(f, \Omega, y) = \deg(f, \Omega_1, y)$.

iii. *Let V be an open and bounded set of $[0, 1] \times \mathbb{R}^m$, $V(t) := \{x \in \mathbb{R}^m : (t, x) \in V\}$ and $F : \overline{V} \to \mathbb{R}^n$, with $f_t = F(t, .) \in C^1(\overline{V}(t))$. Suppose that* *there exists a continuous path $t \to p_t$ such that $p_t \notin f_t(\partial V(t))$, then $\deg(f_t, V(t), p_t)$ is constant in $t \in [0,1]$.*

We need also the following.

Proposition 2. *In a compact set, each connected component is the intersection of all open and closed sets that contain it.*

Proof. See the appendix.

$\square$

In the sequel, we will suppose that the integer n is even. The aim of this note is to provide a variant proofs of the hairy ball theorem. First, we recall the hairy ball theorem.

Theorem 1. (Hairy ball theorem) *An even dimensional sphere does not admit any continuous field of non-zero tangent vectors.*

In other terms, if $g : S \to S$ is continuous and for every $x \in S$, we have $g(x).x = 0$, then there exists $\overline{x}$ such that $g(\overline{x}) = 0$.
In what follows, we establish two equivalent versions of the hairy ball theorem presented as fixed point theorems.

3 Equivalent versions

In the following, we state the first equivalent version to the hairy ball theorem.

Theorem 2. *If $f : S \to S$ is continuous, then either f or $-f$ possesses a fixed point.*

As a first step, we prove that Theorem 2 is equivalent to the hairy ball theorem.

Proof. First, we claim that the hairy ball theorem implies Theorem 2. Indeed, let $f : S \to S$ be a continuous function and consider $g : S \to S$ given by $g(x) = f(x) - (x.f(x))x$. By the hairy ball theorem, g has a zero $\overline{x}$ on S. That is, $f(\overline{x}) = (\overline{x}.f(\overline{x}))\overline{x}$. Now, using that $f(\overline{x})$ is collinear to $\overline{x}$ and that both of them belongs to the sphere, we conclude that either $f(\overline{x}) = \overline{x}$ or $f(\overline{x}) = -\overline{x}$.
Conversely, let $g : S \to S$ be a continuous function such that for any $x \in S$, $g(x).x = 0$. Then for $x \in S$, $g(x) + x \neq 0$. So, we consider the function $f(x) = \frac{x+g(x)}{\|x+g(x)\|}$. By Theorem 2, there exists $\overline{x}$ such that $f(\overline{x}) = \epsilon\overline{x}$, with $\epsilon \in \{-1, 1\}$. This implies that $g(\overline{x}) = 0$. $\square$

Now, we state the main result and we prove that it is equivalent to the hairy ball theorem.

Theorem 3. *If $f : S \to S$ is continuous and satisfying for any $x \in S$, $f(x).x \geq \frac{1}{2}$, then it possesses a fixed point.*

Proof. Obviously, Theorem 2 implies Theorem 3. In fact, since $\|x\| = 1$ on S, then it is trivial to see that $f(x) = -x$ is impossible. Conversely, we will prove that Theorem 3 implies the hairy ball theorem. Indeed, let $g : S \to S$ be a non easy continuous function such that for any $x \in S$, $g(x).x = 0$. Let $M = \sup_{x \in S} \|g(x)\|$. So, put $\alpha = \frac{\sqrt{3}}{M}$ and consider the function $f_\alpha(x) = \frac{x+\alpha g(x)}{\|x+\alpha g(x)\|}$. We have $f_\alpha(x).x = \frac{1}{\|x+\alpha g(x)\|} > 0$ and by simple calculus, we obtain

$$(f_\alpha(x).x)^2 = \frac{1}{\|x + \alpha g(x)\|^2} = \frac{1}{\|x\|^2 + \alpha^2 \|g(x)\|^2} \geq \frac{1}{1 + \alpha^2 M^2} = \frac{1}{4}.$$

By Theorem 3, f_α possesses a fixed point $\overline{x}$. Setting $f_\alpha(\overline{x}).\overline{x} = 1$ above implies that $g(\overline{x}) = 0$, and the result follows. □

Remark 1. *Let us observe that the choice of the real $\frac{1}{2}$ is arbitrary. The main idea of the theorem is that we can allow a radial component if it is not fully opposite to x. Here we state the general version.*
For any real $\lambda \in (-1, 1)$, we denote by P_λ the following proposition
If $f : S \to S$ is continuous and for any $x \in S$, $f(x).x \geq \lambda$, then f has a fixed point.

It is not difficult to prove that P_λ is equivalent to Theorem 2.
To sum up, we have Theorem 3 is equivalent to the hairy ball theorem. Thereafter, providing a proof of Theorem 3 will enable us to have a new proof of the hairy ball theorem which differs from the classical proofs [8].

4 Proof of Theorem 3

The proof of Theorem 3 will depend on the two following results.

Lemma 1. *If $f : S \to S$ is continuous such that for any $x \in S$, $f(x).x \geq \frac{1}{2}$, then there exists $F \in \mathscr{H}$ such that $F(1,.) = f$.*

Theorem 4. *There exists a connected component Γ subset of C_F such that $\Gamma \cap (\{0\} \times S) \neq \emptyset$ and $\Gamma \cap (\{1\} \times S) \neq \emptyset$. Consequently, $F(1,.)$ has a fixed point.*

Once we prove Lemma 1 and Theorem 4, then Theorem 3 is deduced immediately.
Proof of Lemma 1
For any $(t, x) \in [0, 1] \times S$, consider the function F given by

$$F(t, x) = \frac{tf(x) + (1-t)\alpha(x)}{\|tf(x) + (1-t)\alpha(x)\|}. \tag{1}$$

Before introducing the function α, we can easily remark that F satisfies the conclusion of Lemma 1 provided that for any $(t, x) \in [0, 1] \times S$, the function α met the three following properties

(P_1) $tf(x) + (1-t)\alpha(x) \neq 0$.

(P_2) α is continuous.

(P_3) "$\alpha(x)$ is positively collinear to x " is equivalent to " $x = \pm x_0$".

Now, we will construct gradually the function α. First, let us define the function $\beta : S \to S$ by $\beta(x) = y$, where $x = (x_1, \cdots, x_{n+1})$ and $y = (y_1, \cdots, y_{n+1})$ such that

$$\forall i \in \{1, \cdots, n\}\,,\ y_i = x_i\sqrt{x_1^2 + \cdots + x_n^2} = x_i\sqrt{1 - x_{n+1}^2},$$

and

$$y_{n+1} = x_{n+1}\sqrt{2 - x_{n+1}^2}.$$

Second, consider the following rotation R_θ of $S \subset \mathbb{R}^{n+1} = \mathbb{R}^{2k+1}$, north-south axis, whose matrix is given by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & \cdots & 0 & 0 \\ \sin\theta & \cos\theta & \vdots & \vdots & \\ \vdots & & \cos\theta & -\sin\theta & \vdots \\ 0 & & \sin\theta & \cos\theta & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

where $0 < \theta < \frac{\pi}{2}$.

Finally, we define the function $\alpha : S \to S$ by $\alpha(x) = R_\theta(\beta(x))$.

Next, our objective is to prove that the properties (P_1), (P_2) and (P_3) are satisfied. In order to prove (P_1), it suffices to prove that $(tf(x) + (1-t)\alpha(x)).x > 0$. Moreover, since $f(x).x > \frac{1}{2}$, then we just need to prove that $\alpha(x).x > 0$. This follows from the expression of α. Indeed, we have

$$\alpha(x) = R_\theta(\beta(x)) = \begin{pmatrix} \sqrt{1 - x_{n+1}^2}(x_1\cos\theta - x_2\sin\theta) \\ \sqrt{1 - x_{n+1}^2}(x_1\sin\theta - x_2\cos\theta) \\ \sqrt{1 - x_{n+1}^2}(x_3\cos\theta - x_4\sin\theta) \\ \sqrt{1 - x_{n+1}^2}(x_3\sin\theta - x_4\cos\theta) \\ \vdots \\ \sqrt{1 - x_{n+1}^2}(x_{n-1}\cos\theta - x_n\sin\theta) \\ \sqrt{1 - x_{n+1}^2}(x_{n-1}\sin\theta - x_n\cos\theta) \\ x_{n+1}\sqrt{2 - x_{n+1}^2} \end{pmatrix}$$

Therefore, we obtain that

$$\alpha(x).x = (1 - x_{n+1}^2)^{\frac{3}{2}}\cos\theta + x_{n+1}^2\sqrt{2 - x_{n+1}^2} \geq (1 - x_{n+1}^2)^{\frac{3}{2}}\cos\theta + x_{n+1}^2 \geq \min(\frac{2}{3}, \cos\theta) > 0.$$

On the other hand, since all the components of the function β are continuous, then (P_2) is trivial. This completes the proof of (P_1) and (P_2). In order to prove (P_3), note that if $\alpha(x) = \lambda x$ then, $|\lambda| = 1$. Thus, we have $x_{n+1}\sqrt{2-x_{n+1}^2} = \lambda x_{n+1}$. Now, since $\beta(x) = \lambda R_{-\theta}(x)$ and in virtue of [9], the only fixed points of $R_{-\theta}$ are $\pm x_0$, then $x_{n+1} \neq 0$. Therefore, we obtain that $\sqrt{2-x_{n+1}^2} = \lambda = 1$. Squaring the last equality implies that $x = \pm x_0$, as required.

□

In the next section, we will interpret geometrically the properties of the function α.

4.1 The geometrical properties of the function α

For any $\overline{x}_{n+1} \in [-1, 1]$, let us denote by $P_{\overline{x}_{n+1}}$, the following set $P_{\overline{x}_{n+1}} := \{x \in S \text{ such that } x_{n+1} = \overline{x}_{n+1}\}$. By analogy with the unit sphere of $\mathbb{R}^3$, we call this set a parallel. Remark that except at the poles, it is a sphere of dimension $n-1$.

- The image by α of a parallel of altitude x_{n+1} is a parallel of altitude $x_{n+1}\sqrt{2-x_{n+1}^2}$ and closer to the corresponding pole.
- The image by α of a polar cap of altitude x_{n+1} is a polar cap of altitude $x_{n+1}\sqrt{2-x_{n+1}^2}$ and closer to the corresponding pole.

Therefore, the parallel of altitude -1 (reduced to the south pole), of altitude 0 (reduced to the equator) and of altitude 1 (reduced to the north pole) are the only one that are globally invariants. In addition, since $\alpha(x)$ and x are on the sphere and belongs both either to the north semi-sphere or to the south semi-sphere, then co-linearity means equality. As a consequence of the computation of the image of a parallel, we have the following.

Proposition 3. *For any $0 < \mu < \frac{1}{2}$, we have*

$$\alpha(\overline{B}(x_0, \mu) \cap S) \subset \overline{B}(x_0, \frac{\mu}{2}) \cap S.$$

Proof. See the appendix. □

The following section provides the proof of Theorem 4.

4.2 Existence of fixed points

In order to prove Theorem 4, we willneed the following theorem of Mas-Colell [6]. Let X be an open subset of $\mathbb{R}^n$. Let $A \subset X$ be open and such that $\overline{A} \subset X$ and let $\mathbb{F}$ be the set of twice continuously differentiable functions: $F : [0, 1] \times X \to A$.

Theorem 5 (Mas-Colell). *There is an open and dense set $\mathbb{F}' \subset \mathbb{F}$ such that for every $F \in \mathbb{F}'$, any non empty component Γ of C_F with $\Gamma \cap (\{0\} \times X) \neq \emptyset$ is diffeomorphic to a segment.*

The proof of this theorem is based on a transversality argument which make his result only generic. In addition, it is important to notice that Mas-Colell established his result on a convex set X. However, a careful reading of the proof shows that this assumption was only used in a subsequent part.
In order to prove Theorem 4, we need first to prove the following result.

Proposition 4. *Let* $X = B(S, \frac{1}{2})$*,* $A = B(S, \frac{3}{8})$*,* $p \geq 2$ *and* F *given by(1) of Lemma 1. Then there exists an open dense set* $\mathbb{F}' \subset \mathbb{F}$ *and a function* $\widetilde{G}_p : [0,1]\times X \to A$ *such that* $\widetilde{G}_p \in \mathbb{F}'$ *and* $\left\|\widetilde{G}_p(t,x) - F(t, \frac{x}{\|x\|})\right\| \leq \frac{1}{p}$*, for any* $(t,x) \in [0,1] \times X$*.*
Moreover, there exists a connected component $W_p \subset C_{\widetilde{G}_p}$ *such that* $W_p \cap (\{0\} \times X) \neq \emptyset$ *and* $W_p \cap (\{1\} \times X) \neq \emptyset$*.*

Proof. In order to apply Theorem 5, we present here a constructive proof.
Let F be the function given by (1) and consider the function $\widetilde{F} : [0,1] \times \overline{X} \to S$ defined by $\widetilde{F}(t,x) = F(t, \frac{x}{\|x\|})$. By Stone Weierstrass approximation method, for any integer $p \geq 2$, there exists a C^2 function $\widetilde{F}_p : [0,1] \times \overline{X} \to A$, such that $\left\|\widetilde{F} - \widetilde{F}_p\right\|_\infty \leq \frac{1}{2p}$. Let $\mathbb{F}'$ be an open dense set given by Mas-Colell's theorem such that $\overline{\mathbb{F}'} = \mathbb{F}$. Since $\widetilde{F}_p \in \mathbb{F}$, then there exists $\widetilde{G}_p \in \mathbb{F}'$ such that $\widetilde{G}_p : [0,1] \times X \to A$ and $\left\|\widetilde{G}_p - \widetilde{F}_p\right\|_\infty \leq \frac{1}{2p}$. So, we obtain that

$$\left\|\widetilde{G}_p - \widetilde{F}\right\|_\infty \leq \frac{1}{p}. \tag{2}$$

It remains to show that $C_{\widetilde{G}_p} \neq \emptyset$. Indeed, let us prove that for any $0 < r < \frac{1}{3}$, there exists $x_p \in \overline{B}(x_0, r) \subset X$, such that $\widetilde{G}_p(0, x_p) = x_p$.
By inequality (2), we have $\left\|\widetilde{G}_p(0,x) - \widetilde{F}(0,x)\right\| = \left\|\widetilde{G}_p(0,x) - \alpha(\frac{x}{\|x\|})\right\| \leq \frac{1}{p}$. This implies that for each $x \in \overline{B}(x_0, r)$, we have

$$\left\|\widetilde{G}_p(0,x) - x_0\right\| \leq \left\|\widetilde{G}_p(0,x) - \alpha(\frac{x}{\|x\|})\right\| + \left\|\alpha(\frac{x}{\|x\|}) - x_0\right\| \leq \frac{1}{p} + \left\|\alpha(\frac{x}{\|x\|}) - x_0\right\|.$$

A simple calculus*, shows that for any $x \in \overline{B}(x_0, r)$, we have $\alpha(\frac{x}{\|x\|}) \in \overline{B}(x_0, \frac{r}{\sqrt{2}\sqrt{1+\sqrt{1-r^2}}})$. Therefore, by Proposition 3, we conclude that $\widetilde{G}_p$ maps $\overline{B}(x_0, r)$ into itself and by Brouwer theorem, there exists $x_p \in \overline{B}(x_0, r)$ such that $x_p \in C_{\widetilde{G}_p}$.
At this step, applying Theorem 5 to $\widetilde{G}_p$, we conclude that any component $\Gamma_p \in C_{\widetilde{G}_p}$ starting from $(0, x_p)$ is diffeomorphic to a segment. Now, it remains to show that there exists a component Γ_p such that $\Gamma_p \cap (\{1\} \times X) \neq \emptyset$. First, it is important to notice that for any component in $C_{\widetilde{G}_p}$, we have either it starts from 0 and belongs to the family denoted by $\Gamma_{p,i}$, or it doesn't intercept $\{0\} \times$

*The proof is related in the appendix

X and in this case belongs to the second family denoted by $\Gamma'_{p,i}$. Thus, we have $CC_{\widetilde{G}_p} = \{\Gamma_{p,i}\}_{i\in I_1} \cup \{\Gamma'_{p,i}\}_{i\in I_2}$, where $CC_{\widetilde{G}_p}$ is the set of all components of $C_{\widetilde{G}_p}$. Now, let $\overline{G}_p$ be the translation of $\widetilde{G}_p$. In the sequel, we abuse notation by writting $C_{\widetilde{G}_p} = C_{\overline{G}_p} := \{(t,x) \in [0,1] \times X \text{ such that } \overline{G}_p(t,x) = 0\}$. We argue by contradicition. Suppose that there is no component $\Gamma_{p,i}$ intersecting $(\{1\} \times X)$, then there exists $t_i < 1$ such that $\Gamma_{p,i} \subset [0,t_i) \times X$. By Proposition 2, we have $\Gamma_{p,i} = \cap_{j\in J} U_{i,j}$, where for any $j \in J$, $U_{i,j}$ is open and closed in $C_{\widetilde{G}_p}$. Let $D = C_{C_{\widetilde{G}_p}} \backslash ([0,t_i) \times X)$, then $(\cap_{j\in J} U_{i,j}) \cap D = \emptyset$. Since D is compact and the family $U_{i,j}$ has the finite intersection property, we have that $(\cap_{j\in J_1 \subset J} U_{i,j}) \cap D = \emptyset$, for J_1 finite. Moreover, note that $U = \cap_{j\in J_1} U_{i,j}$ is open and closed set containing $\Gamma_{p,i}$, then there exists $j_0 \in J_1$ such that $U = U_{i,j_0}$ included in $[0,t_i) \times X$.
For the second type of components $\Gamma'_{p,i}$ of $C_{\widetilde{G}_p}$, by a similar argument, there exists $j'_0 \in J_2$ finite such that $U = U'_{i,j'_0}$ open and closed included in $(r_i, 1] \times X$.
The family $\left\{(U_{i,j_0})_{i\in I_1}, (U'_{i,j'_0})_{i\in I_2}\right\}$ forms an open covering of the compact set $C_{\widetilde{G}_p}$. Therefore, we can extract a finite sub-covering $\{U_1, \cdots, U'_n\}$ such that the first k components are included in $[0,\overline{t}[\times X$ and the remaining $(n-k)$ components are included in $(\overline{r}, 1] \times X$, where $\overline{t} = \max(t_i)_{1\le i\le k}$ and $\overline{r} = \min(r_i)_{k+1\le i\le n}$.

The set $E = \cup_{i\le k} U_i$ is open and closed in the compact set $C_{\widetilde{G}_p}$. So E is compact and contained in $[0,\overline{t}) \times X$. Moreover, $E^c = C_{C_{\widetilde{G}_p}} \backslash E$ is also compact, open and contained in $(\overline{r}, 1] \times X$. We have $C_{\widetilde{G}_p} \subset \mathrm{Im}(\widetilde{G}_p) \subset A \subset \Omega = B(S, \frac{7}{16}) \subset X$. Using the separation criteria in the Hausdorff metric space $[0,1] \times \Omega$, there exists two disjoint open sets V_1 and V_2 in $[0,1] \times \Omega$ such that $E \subset V_1$, and $E^c \subset V_2$. The rest of the proof is ruled out in four steps collected in the following lemma.

Lemma 2. *We have*

1. $\deg(\overline{G}_p(0,.), \Omega, 0) = \deg(\overline{F}(0,.), \Omega, 0)$.
2. $\deg(\overline{G}_p(0,.), \Omega, 0) = -2$.
3. $\deg(\overline{G}_p(0,.), \Omega, 0) = \deg(\overline{G}_p(0,.), V_1(0), 0)$.
4. $\deg(\overline{G}_p(t,.), V_1(t), 0)$ *is constant in* $t \in [0,1]$.

Assuming this Lemma through, we claim that there exists a component $W_p \subset C_{\overline{G}_p}$ such that $W_p \cap (\{0\} \times X) \neq \emptyset$ and $W_p \cap (\{1\} \times X) \neq \emptyset$. Indeed, by Lemma 2 (4), we have

$$\deg(\overline{G}_p(t,.), V_1(t), 0) = \deg(\overline{G}_p(0,.), V_1(0), 0) = -2.$$

Yet, by construction for t close enough to 1, we have $V_1(t) = \emptyset$. Therefore, $\deg(\overline{G}_p(t,.), V_1(t), 0) = 0$, which establish a contradiction, as required. $\square$

The proof of Lemma 2 is based on the degree properties and will take the rest of this section.

Proof. of Lemma 2

1. Let f and g given by the following: for any $x \in \overline{\Omega}$, $f(x) = \overline{F}(0,x) = \widetilde{F}(0,x) - x = \alpha(\frac{x}{\|x\|}) - x$ and $g(x) = \overline{G}_p(0,x)$. We claim that $0 \notin f(\partial\Omega)$. Indeed, if there exists $x \in \partial\Omega$ such that $f(x) = 0$, then x is a fixed point of α. That is, $x = \pm x_0 \notin \partial\Omega$, which yields a contradiction. On the other hand, for any $x \in \overline{\Omega}$, we have $\left\|\widetilde{G}_p(0,x) - \alpha(\frac{x}{\|x\|})\right\| \leq \frac{1}{p}$. That is, $\|g(x) - f(x)\| \leq \frac{1}{p} \leq r$, where $r = d(0, f(\partial\Omega)) > 0$. By Proposition 1 (i), we have $\deg(\overline{G}_p(0,.), \Omega, 0) = \deg(\overline{F}(0,.), \Omega, 0)$.
2. It suffices to prove that $\deg(\overline{F}(0,.), \Omega, 0) = -2$. We recall that

$$\deg(\overline{F}(0,.), \Omega, 0) = \sum_{x \in \overline{F}^{-1}(0,.)(\{0\})} \operatorname{sgn}(\det\ D\overline{F}(0,x)) = \sum_{x \in \{x_0, -x_0\}} \operatorname{sgn}(\det\ D\overline{F}(0,x)).$$

Next, we compute the differential of $\overline{F}(0,.)$ at the points x_0 and $-x_0$. Define $h : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}^{n+1}$ by $h(x) = \beta(\frac{x}{\|x\|})$, then we have for $x \neq 0$, $D\overline{F}(0,.)(x) = R_\theta(Dh(x)) - I_{n+1}$.
Let us denote by $x = (x', x_{n+1})$, we have

$$h(x') = \frac{\|x'\|\, x'}{\|x\|^2},$$

and

$$(h(x))_{n+1} = \frac{x_{n+1}\sqrt{2\,\|x'\|^2 + x_{n+1}^2}}{\|x\|^2}.$$

The function h is differentiable at x_0 and $Dh(x_0) = 0$. Indeed, we have $h(x) - h(x_0) = (\frac{\|x'\|x'}{\|x\|^2}, \frac{x_{n+1}(x_{n+1}^2 + 2\|x'\|^2)^{\frac{1}{2}}}{\|x\|^2} - 1)$.
It suffices to prove that $h(x) - h(x_0) = o(\|x - x_0\|)$. Since we have $\|x - x_0\|^2 = \|x'\|^2 + (x_{n+1} - 1)^2$, then we have only to prove that every component belongs to $o(\|x'\|)$. It is clear that $\frac{\|x'\|x'}{\|x\|^2} = o(\|x'\|)$. On the other hand, for the second component of $h(x) - h(x_0)$, we have

$$\frac{x_{n+1}(x_{n+1}^2 + 2\,\|x'\|^2)^{\frac{1}{2}}}{\|x\|^2} - 1 = \frac{x_{n+1}^2(1 + \frac{2\|x'\|^2}{x_{n+1}^2})^{\frac{1}{2}} - \|x\|^2}{\|x\|^2}$$

$$= \frac{x_{n+1}^2(1 + \frac{\|x'\|^2}{x_{n+1}^2} + o(\frac{\|x'\|^2}{x_{n+1}^2})) - \|x'\|^2 - x_{n+1}^2}{\|x\|^2} = \frac{o(\|x'\|^2)x_{n+1}^2}{\|x\|^2}.$$

Hence, h is differentiable at x_0 and $Dh(x_0) = 0$. A similar calculus leads to $Dh(-x_0) = 0$. Finally, we can conclude that $D\overline{F}(0,.)x_0 = D\overline{F}(0,.)(-x_0) = -I_{n+1}$. Thus, we have $\deg(\overline{F}(0,.), X, 0) = (-1)^{n+1} + (-1)^{n+1} = -2$, since n is even, and the result follows.

3. We recall that $V_1(t) = \{x \in \Omega : (t,x) \in V_1\}$ and let $\Omega_1 = V_1(0)$. We have $0 \notin \overline{G}_p(\overline{\Omega}\backslash V_1(0))$. Indeed, suppose that there exists $x \in \overline{\Omega}\backslash V_1(0)$ such that $\overline{G}_p(0,x) = 0$. That is, $x \notin V_1(0)$ and $(0,x) \in C_{\widetilde{G}_p} \subset V_1 \cup V_2$. This implies that $(0,x) \in V_2$. Though, by construction, V_2 doesn't contain $(0,x)$ which set a contradiction. By Proposition 1 (ii), we get $\deg(\overline{G}_p(0,.), \Omega, 0) = \deg(\overline{G}_p(0,.), V_1(0), 0)$.

4. Let $V(t) = V_1(t)$, $f_t = \overline{G}_p(t,.)$ and $p_t = 0$. We have $0 \notin f_t(\partial V_1(t))$. Suppose that there exists $x \in \partial V_1(t)$ such that $\overline{G}_p(t,x) = 0$, then by definition $(t,x) \in \partial V_1 = \overline{V}_1\backslash V_1$ and $(t,x) \in C_{\widetilde{G}_p} = E \cup E^c$. Thus, we have two cases. If $(t,x) \in E$, then $(t,x) \in V_1$, an impossibility since $(t,x) \in V_1^c$. Otherwise, we have $(t,x) \in E^c$, then by construction of V_1 and V_2, we have $E^c \cap \partial V_1 = \emptyset$, contradiction. Hence, by By Proposition 1 (iii), we conclude that $\deg(\overline{G}_p(t,.), V_1(t), 0)$ is constant in $t \in [0,1]$.

□

Transition smooth-continuous version

To sum up, the result holds for the functions twice continuously differentiable. In this section, we will extend our result to continuous functions and recover the fixed points of F. Now, we come to the heart of the proof of Theorem 4.

Proof. From the above result, there exists x_p such that $(0,x_p) \in W_p \cap (\{0\} \times X) \subset C_{\widetilde{G}_p}$. Since the sequence x_p is bounded, then we may assume that it converges to some $\overline{x}$. Thus, we have $\lim_{p \to +\infty} \overline{G}_p(0,x_p) = \overline{F}(0,\overline{x}) = \alpha(\frac{\overline{x}}{\|\overline{x}\|}) - \overline{x} = 0$. As explained before, this yields to $\overline{x} = \pm x_0$. Without loss of generality, we can suppose that the sequence $(x_p)_{p \geq 0}$ converges to x_0. We denote by W_k^{tr}, the translated component of W_k, given by $W_k^{tr} = W_k + (x_0 - x_k)$. Thus, in the spirit of the lim sup, we put $Z = \cap_{p>1}\overline{\cup_{k \geq p} W_k^{tr}} = \cap_{p>1} Z_p$.
We prove that Z is connex. Since the components W_k^{tr} are connex, for all k, and contains a common point $(0,x_0)$, then $\cup_{k \geq p} W_k^{tr}$ is connex. Therefore, the set $Z_p = \overline{\cup_{k \geq p} W_k^{tr}}$ is the closure of a compact connex set. Hence, Z is connex.
Now, we claim that $Z \subset C_F$. Let $(t,x) \in Z$, then for any p, we have $(t,x) \in Z_p$. That is $B((t,x), \frac{1}{p}) \cap (\cup_{k \geq p} W_k^{tr}) \neq \emptyset$. By definition, we obtain that $\exists\ k(p) \geq p$, such that $(t_p, z_p) \in B((t,x), \frac{1}{p}) \cap W_{k_{(p)}}^{tr}$. That is, $(t_p, z_p) \to (t,x)$ and $(t_p, z_p) \in W_{k(p)}^{tr}$. Let $z'_p = z_p - (x_0 - x_{k(p)})$, then $(t_p, z'_p) \in W_{k(p)} \subset C_{\widetilde{G}_{k(p)}}$. Equivalently, we have $\widetilde{G}_{k(p)}(t_p, z'_p) = z'_p$. Since $x_{k(p)}$ converges to x_0, then we have that $(t_p, z'_p) \to (t,x)$. By continuity of $\widetilde{F}$, we obtain that $d(\widetilde{F}(t,x), \widetilde{F}(t_p, z'_p)) < \frac{\epsilon}{4}$. We have already established that for any p large enough, $\left\|\widetilde{F}(t_p, z'_p) - \widetilde{G}_{k(p)}(t_p, z'_p)\right\|_\infty \leq \frac{1}{p} < \frac{\epsilon}{4}$. This implies that, $d(\widetilde{F}(t,x), \widetilde{G}_{k(p)}(t_p, z'_p)) = d(\widetilde{F}(t,x), z'_p) < \frac{\epsilon}{2}$. Hence, we conclude that $d(\widetilde{F}(t,x), x) \leq d(\widetilde{F}(t,x), z'_p) + d(z'_p, x) \leq \epsilon$. Tending ϵ towards zero, we obtain that $\widetilde{F}(t,x) = x$. In addition, we have both C_F and $C_{\widetilde{F}}$ are included in S. Since $C_{\widetilde{F}} \subset \mathrm{Im}(\widetilde{F}) = \mathrm{Im}(F) \subset$

S and F and $\widetilde{F}$ coincide on S, then $C_{\widetilde{F}} = C_F$, as required.
It remains to show that $Z \cap (\{1\} \times S) \neq \emptyset$. Using our previous result, we know that for any $k \geq p$, there exists $(1, x'_k) \in W_k$. Let ${x''}^{tr}_k = x'_k + (x_0 - x_k)$. We have ${x''}^{tr}_k \in W^{tr}_k \subset \cup_{j \geq p} W^{tr}_j \subset Z_p$. Since Z_p is compact, then $\lim_{k \to +\infty} {x''}^{tr}_k = y$ belongs to Z_p. On the other hand, we have $\widetilde{G}_k(1, x'_k) = x'_k$. Passing to the limit leads to $\widetilde{F}(1, y) = y \in S$. Therefore, we have $(1, y) \in Z_p \cap (\{1\} \times S)$. Consequently, for any finite set J, we have that $\cap_{p \in J} Z_p \cap (\{1\} \times S) \neq \emptyset$. Thus, applying the finite intersection property, we obtain that $Z \cap (\{1\} \times S) \neq \emptyset$ proving the main result. $\square$

5 Appendix

i. *Proof of Proposition 2*

Let C be a connected component of the compact set K and $(K_i(C))_{i\in I}$ be the family of all open and closed sets of K that contains C. We denote by $K(C) = \cap_{i\in I} K_i(C)$. We have, $K(C)$ is closed in the compact set K, then compact. We prove that $C = K(C)$. We have $C \subset K_i(C)$, for any $i \in I$. Indeed, since $K_i(C)^c$ is also open and closed, then if $C \cap K_i(C)^c \neq \emptyset$, this will contradicts that C is connex. Therefore, $C \subset K(C)$.

Conversely, it suffices to prove that $K(C)$ is connex. We argue by contradiction. Suppose that $K(C) = F_1 \cup F_2$, where F_1 and F_2 are nonempty, open, closed and disjoints sets. Using the separation criteria, we obtain that there exists U_1, U_2 two disjoints open sets of K such that $F_1 \subset U_1$ and $F_2 \subset U_2$. Since $C \subset K(C)$, then suppose that $C \subset F_1 \subset U_1$. Let $U = U_1 \cup U_2$, then $K(C) \cap U^c = \emptyset$. That is, $\cap_{i\in I} K_i(C) \cap U^c = \emptyset$. Using the finite intersection property, we obtain that there exists I_1 finite such that $\cap_{i\in I_1\subset I} K_i(C) \cap U^c = \emptyset$. Therefore, there exists $i_0 \in I_1$, such that $K_{i_0} \subset U$. However, $K_{i_0} \cap U_1$ is open and closed in K containing C but not $K(C)$, which establish a contradiction.

□

ii. *Proof of Proposition 3*

Let $x \in \overline{B}(x_0, \mu) \cap S$, then $\|x - x_0\|^2 = 2(1 - x_{n+1}) \le \mu^2$.

On the other hand, we have $\|\alpha(x) - x_0\|^2 = 2(1 - x_{n+1}\sqrt{2 - x_{n+1}^2})$
$= \frac{2(1-x_{n+1}^2(2-x_{n+1}^2))}{1+x_{n+1}\sqrt{2-x_{n+1}^2}} = \frac{2(1-x_{n+1}^2)^2}{1+x_{n+1}\sqrt{2-x_{n+1}^2}} \le \frac{2(1-x_{n+1})^2(1+x_{n+1})^2}{1+x_{n+1}}$
$= 2(1 - x_{n+1})^2(1 + x_{n+1}) \le 4(1 - x_{n+1})^2 \le \mu^4 \le \frac{\mu^2}{4}$, for any $0 < \mu < \frac{1}{2}$.

□

iii. Let $0 < r < \frac{1}{3}$, then for any $x \in \overline{B}(x_0, r)$, we have $\alpha(\frac{x}{\|x\|}) \in \overline{B}(x_0, \frac{r}{\sqrt{2}\sqrt{1+\sqrt{1-r^2}}})$.

Proof. Let $x \in \overline{B}(x_0, r)$, then $\|x - x_0\|^2 = \|x\|^2 - 2x_{n+1} + 1 \le r^2$. Let $t = x_1^2 + \cdots + x_n^2$ and $s = x_{n+1}$, then $t + s^2 - 2s + 1 = (1 - s)^2 + t \le r^2$. On the other hand, we have $\left\|\frac{x}{\|x\|} - x_0\right\|^2 = 2(1 - \frac{x_{n+1}}{\|x\|}) = 2(1 - \frac{s}{\sqrt{t+s^2}})$. Put $f(t) = 2(1 - \frac{s}{\sqrt{t+s^2}})$ for any $t \in (0, r^2 - (1 - s)^2)$, then f is increasing and $f(t) \le f(r^2 - (1 - s)^2) = 2(1 - \frac{s}{\sqrt{r^2-1+2s}}) = g(s)$. Nown the function g is defined on $(1 - r, 1)$ and g reaches its maximum at $(1 - r^2)$, then $g(s) \le g(1 - r^2) = 2(1 - \sqrt{1 - r^2}) = \frac{2r^2}{1+\sqrt{1-r^2}}$. Hence, we obtain that $\left\|\frac{x}{\|x\|} - x_0\right\| \le \frac{\sqrt{2}r}{\sqrt{1+\sqrt{1-r^2}}} < \frac{1}{2}$. By Proposition 3, we have

$\alpha(\frac{x}{\|x\|}) \in \overline{B}(x_0, \frac{r}{\sqrt{2}\sqrt{1+\sqrt{1-r^2}}})$, as required. □

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